Second-order light deflection by tidal charged black holes on the brane

László Árpád Gergely[1,2⋆], **Zoltán Keresztes**[1,2¶], **Marek Dwornik**[1,2†]

[1] Department of Theoretical Physics, University of Szeged, Tisza Lajos krt 84-86, Szeged 6720, Hungary

[2] Department of Experimental Physics, University of Szeged, Dóm tér 9, Szeged 6720, Hungary

⋆ gergely@physx.u-szeged.hu, ¶zkeresztes@titan.physx.u-szeged.hu,
† marek@titan.physx.u-szeged.hu

Abstract. We derive the deflection angle of light rays passing near a black hole with mass m and tidal charge q, confined to a generalized Randall-Sundrum brane with codimension one. We employ the weak lensing approach, up to the second order in perturbation theory by two distinct methods. First we adopt a Lagrangian approach and derive the deflection angle from the analysis of the geodesic equations. Then we adopt a Hamiltonian approach and we recover the same result from the analysis of the eikonal. With this we re-establish the unicity of the result as given by the two methods. Our results in turn implies a more rigurous constraint on the tidal charge from Solar System measurements, then derived before.

1. Introduction

The possibility that the gravitational interaction acts in a more than four-dimensional non-compact space-time [1], while keeping the other interactions locked in four space-time dimensions, has raised interesting new perspectives in the solvability of the hierarchy problem and in cosmological evolution [2]. This hypothesis has led to alternative explanations for dark matter [3], [4] and [5].

The most common such brane-world model is five-dimensional, containing a four-dimensional (time-evolving three-dimensional) brane. In the context of such generalized Randall-Sundrum brane-worlds, both the five-dimensional space-time and the embedded brane are allowed to have generic curvature. Gravitational dynamics on the brane is governed by an effective Einstein equation [6], derived in full generality in [7], [8]. Higher codimension branes were also considered in connection with conical singularities [9], [10].

Spherically symmetric brane-world black holes were studied both by analytical and numerical methods. Six-dimensional vacuum black holes, which are locally higher dimensional Schwarzschild, were found in Ref. [11]. A shell-like distribution of five-dimensional matter (violating both the weak and strong energy conditions in the vicinity of the brane, but falling off at infinity) is able to support a static five-dimensional black hole localized on the brane, with the horizon rapidly "decaying" in the extra dimension, [12], [13]. Black holes with a radiating component in the extra dimension were investigated in connection with the AdS/CFT correspondence [14]. A numerical analysis in an asymptotically five-dimensional Anti de Sitter space-time showed the existence of small brane black holes (compared to the five-dimensional curvature) as five-dimensional Schwarzschild solutions [15], however at large masses the effect of the brane forbids to find such solutions. The generic static, spherically symmetric, five-dimensional vacuum black hole on a three-dimensional time-evolving brane, as a solution of the full set of the five-dimensional Einstein equations was not found yet.

A static, spherically symmetric, vacuum black hole solution of the effective Einstein equation‡ has been found [16], but the five-dimensional space-time in which it can be embedded, remains unknown. Beside the mass m, this black hole has a *tidal* charge q arising from the Weyl curvature of the higher dimensional space-time, as shown in the line element

$$ds^2 = -f\left(r\right)dt^2 + f^{-1}\left(r\right)dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \ , \tag{1}$$

with the metric function

$$f\left(r\right) = 1 - \frac{2m}{r} + \frac{q}{r^2} \, . \tag{2}$$

Formally the metric (1) is the Reissner-Nordström solution of a spherically symmetric Einstein-Maxwell system in general relativity. There, however the place of the tidal charge q is taken by the square of the *electric* charge Q. Thus $q = Q^2$ is always positive, when the metric (1) describes the spherically symmetric exterior of an electrically charged object in general relativity. By contrast, in brane-world theories the metric (1) allows for any q.

‡ Besides having no matter on the brane, any five-dimensional matter source could be present only if the pull-back of its energy-momentum tensor to the brane vanishes. Also, the embedding of the brane is symmetric.

When $|q| \gg 2mr$ (thus for a light black hole and not too far from its horizon) and $q < 0$ the tidal charge term dominates. This remark is consistent with the result of Ref. [15], according to which five-dimensional Schwarzschild solutions (with $f = 1 - m_5/r^2$) can be considered as brane black holes as long as their mass is small compared to the five-dimensional curvature radius.

On the other hand when $|q| \ll 2mr$, the term containing the tidal charge becomes a mere correction of the four-dimensional Schwarzschild metric. The four-dimensional Schwarzschild metric can be extended into the fifth dimension as a black string [17], which solves the five-dimensional Einstein equations. Gravity wave perturbations of such a black-string brane-world were computed in Ref. [18]. Due to the Gregory-Laflamme instability [19] the string is expected to pinch off, thus a black cigar emerges [20] (although under very mild assumptions, classical event horizons will not pinch off [21]).

For $|q| \ll 2mr$ (small tidal charge and / or far from the horizon) the correction to the Schwarzschild potential represented by the tidal charge term scales with r^{-2}. On the other hand the perturbative analysis of the gravitational field of a spherically symmetric source in the weak field limit in the original Randall-Sundrum setup (Schwarzschild black hole on a brane embedded in Anti de Sitter five-dimensional space-time) has shown corrections to the Schwarzschild potential scaling as r^{-3} [1], [22] and [23]. Therefore the five-dimensional extension of the tidal charged black hole is not a linearly perturbed five-dimensional Anti de Sitter space-time [24], but should rather be considered as a perturbation of the black string / black cigar metric.

The case $q > 0$ is in full analogy with the general relativistic Reissner-Nordström solution. For $q < m^2$ it describes tidal charged black holes with two horizons at $r_h = m \pm \sqrt{(m^2 - q)}$, both below the Schwarzschild radius. For $q = m^2$ the two horizons coincide at $r_h = m$ (this is the analogue of the extremal Reissner-Nordström black hole). In these cases it is evident that the gravitational deflection of light and gravitational lensing is decreased by q. Finally there is a new possibility forbidden in general relativity due to physical considerations on the smallness of the electric charge. This is $q > m^2$ for which the metric (1) describes a naked singularity. Such a situation can arise whenever the mass m of the brane object is of small enough, compared to the effect of the bulk black hole generating Weyl curvature, and as such, tidal charge. Due to its nature, the tidal charge q should be a more or less global property of the brane, which can contain many black holes of mass $m \geq \sqrt{q}$ and several naked singularities with mass $m < \sqrt{q}$.

For any $q < 0$ there is only one horizon, at $r_h = m + \sqrt{(m^2 + |q|)}$. For these black holes, gravity is increased on the brane by the presence of the tidal charge [16]. Light deflection and gravitational lensing are stronger than for the Schwarzschild solution.

The metric (1) can be also considered as the exterior of a star. In this case one does not have to worry about the existence or location of horizons, as they would lie inside the star, where some interior solution should be matched to the metric (1). The generic feature that a positive (negative) tidal charge is weakening (strengthening) gravitation on the brane, is kept.

Tidal charged brane black hole metrics were studied before as vacuum exteriors for interior stellar solution [25], with the purpose of confrontation with solar system tests [26], evolution of thin accretion disks in this geometry [27] and in a thermodynamical context [28].

Gravitational lensing could provide a test of such brane-world solutions, in particular it may turn useful in the study of the tidal charged black hole. The

determination of the tidal charge can indicate ways to extend this solution into the fifth dimension. Both weak [29], [30] and strong [31] gravitational lensing of various five-dimensional black holes were discussed, the topic being reviewed in Ref. [32].

In this paper we derive the deflection angle of light rays caused by brane black holes with tidal charge (1). Generalizing previous approaches [29], [30], we carry on this computation up to the second order in the weak lensing parameters. As the metric (1) is static, we consider only the second order gravielectric contributions, but no gravimagnetic contributions, which are of the same order and would appear due to the movement of the brane black holes. Gravimagnetic effects in the general relativistic approach were considered in [33].

In Section 2 we present a Lagrangian approach, based on Ref. [34]. We conclude this section with the derivation of the light deflection angle to second order accuracy in both m and q. The first order contributions are in agreement with the results of Ref. [35]. The second order contributions however differ from the corresponding result of Ref. [26], obtained by a Hamilton-Jacobi approach. In order to sort out this discrepancy, in Section 3 we carefully employ the eikonal method to the required order. As consequence our previous Lagrangian result is reproduced, re-establishing the unicity of the expression for light deflection. Starting from the improved result, we could strengthen the constraint on the brane tension in Section 4. Section 5 contains the concluding remarks.

2. Lagrangian approach

2.1. Light propagation

Light follows null geodesics of the metric (1). Its equations of motion can be derived either from the geodesic equations, or from the Lagrangian given by $2\mathcal{L} = \left(ds^2/d\lambda^2\right)$, where λ is a parameter of the null geodesic curve (see Chapter 3 of Ref. [36]. Due to spherical and reflectional symmetry across the equatorial plane, $\theta = \pi/2$ can be chosen. Thus

$$0 = 2\mathcal{L} = -f\left(r\right)\dot{t}^2 + f^{-1}\left(r\right)\dot{r}^2 + r^2\dot{\varphi}^2 \,. \tag{3}$$

(A dot represents derivative with respect to λ.) The cyclic variables t and φ lead to the constants of motion E and L

$$E \equiv -p_t = f\dot{t}\,, \qquad L \equiv p_\varphi = r^2\dot{\varphi}\,. \tag{4}$$

By inserting these into Eq. (3), passing to the new radial variable $u = 1/r$ and introducing φ as a dependent variable, we obtain

$$\left(u'\right)^2 = \frac{E^2}{L^2} - u^2 f\ \left(u\right)\,, \tag{5}$$

where a prime refers to differentiation with respect to φ.

Unless $u' = 0$ (representing a circular photon orbit), differentiation of Eq. (5) gives

$$u'' = -uf\ - \frac{u^2}{2}\frac{df}{du}\,, \tag{6}$$

For $f = 1$, when there is no gravitation at all (the metric (1) becomes flat), the above equation simplifies to $u'' + u = 0$, which is solved for $u = u_0 = b^{-1}\cos\varphi$. The impact parameter b represents the closest approach of the star on the straight line orbit obtained by disregarding the gravitational impact of the star (this is the viewpoint

an asymptotic observer will take, as the metric (1) is asymptotically flat). The polar angle φ is measured from the line pointing from the centre of the star towards the point of closest approach. With $u' = 0$ at the point of closest approach, given in the asymptotic limit by $u = b^{-1}$, Eq. (5) with $m = 0 = q$ gives $b = L/E$.

2.2. Perturbative solution

Eq. (6), written in detail, gives

$$u'' + u = 3mu^2 - 2qu^3 \ . \tag{7}$$

For studying weak lensing, we look for a perturbative solution in series of the small parameters

$$\varepsilon = mb^{-1} \,, \qquad \eta = qb^{-2} \tag{8}$$

in the form

$$u = b^{-1}\cos\varphi + \varepsilon u_1 + \eta v_1 + \varepsilon^2 u_2 + \eta^2 v_2 + \varepsilon\eta w_2 + \mathcal{O}\left(\varepsilon^3, \eta^3, \varepsilon\eta^2, \varepsilon^2\eta\right) \ . \tag{9}$$

The index on the unknown functions u_1, u_2, v_1, v_2 and w_2 counts the perturbative order in which they appear. By inserting Eq. (9) into the weak lensing equation (7) we obtain the relevant differential equations for the unknown functions. Up to the second order in both small parameters these are:

$$\varepsilon \ : \qquad u_1'' + u_1 = 3b^{-1}\cos^2\varphi \,, \tag{10}$$

$$\eta \ : \qquad v_1'' + v_1 = -2b^{-1}\cos^3\varphi \,, \tag{11}$$

$$\varepsilon^2 : \qquad u_2'' + u_2 = 3u_1\left[u_1\left(m - 2qb^{-1}\cos\varphi\right) + 2\cos\varphi\right] \,, \tag{12}$$

$$\eta^2 : \qquad v_2'' + v_2 = 3v_1\left[v_1\left(m - 2qb^{-1}\cos\varphi\right) - 2\cos^2\varphi\right] \,, \tag{13}$$

$$\begin{aligned} \varepsilon\eta : \qquad & w_2'' + w_2 = 6\big[u_1 v_1\left(m - 2qb^{-1}\cos\varphi\right) \\ & + v_1\cos\varphi - u_1\cos^2\varphi\big] \,. \end{aligned} \tag{14}$$

Note that the solutions are not allowed to contain contributions with the property $f(-\varphi) = -f(\varphi)$, as the zero of φ was chosen at the point of closest approach $r_{\min}$, with respect to which the past and future portions of the path are symmetric (this is a consequence of the static nature of the lensing metric (1).) The first order equations (10) and (11) are solved for

$$u_1 = b^{-1}\left[C_\varepsilon \cos\varphi + \frac{1}{2}\left(3 - \cos 2\varphi\right)\right] \,, \tag{15}$$

$$v_1 = b^{-1}\left[C_\eta \cos\varphi + \frac{1}{16}\left(\cos 3\varphi - 12\varphi\sin\varphi\right)\right] \,, \tag{16}$$

where $C_{\varepsilon,\ \eta}$ are constants of integration appearing at the order shown by their indices and we have dropped (by choosing as zero their pre-factors, which are constants of integration) the terms proportional to $\sin\varphi$, in accordance with the earlier remark. To keep all terms in u_1 and v_1 of comparable order, we have factored out b^{-1} from the constants. Thus, both mu_1 and mv_1 are of order ε, while both $qb^{-1}u_1$ and $qb^{-1}v_1$ are of order η. In consequence, all these terms drop out from Eqs. (12)-(14), which are then solved for

$$u_2 = b^{-1}\left[C_{\varepsilon^2}\cos\varphi + C_\varepsilon\left(3 - \cos 2\varphi\right) + \frac{3}{16}\left(\cos 3\varphi + 20\varphi\sin\varphi\right)\right] , \tag{17}$$

$$v_2 = b^{-1}\big[C_{\eta^2}\cos\varphi + \frac{3}{16}C_\eta\left(\cos 3\varphi - 12\varphi\sin\varphi\right)$$

$$
\begin{aligned}
&+\frac{1}{256}\big(-21\cos 3\varphi+\cos 5\varphi+60\varphi\sin\varphi \\
&-36\varphi\sin 3\varphi-72\varphi^2\cos\varphi\big)\big]\,, \qquad (18)
\end{aligned}
$$

$$
\begin{aligned}
w_2 = b^{-1}&\left[C_{\varepsilon\eta}\cos\varphi+C_\eta\left(3-\cos 2\varphi\right)+\frac{3}{16}C_\varepsilon\left(\cos 3\varphi-12\varphi\sin\varphi\right)\right. \\
&\left.+\frac{1}{16}\left(-60+31\cos 2\varphi-\cos 4\varphi+12\varphi\sin 2\varphi\right)\right], \qquad (19)
\end{aligned}
$$

where $C_{\varepsilon^2,\ \eta^2,\ \varepsilon\eta}$ represent additional constants of integration (and as before, we have dropped $\sin\varphi$ terms, which also arise by integration). The remaining integration constants can be fixed by inserting the solution (9) with the coefficients (15)-(19) into Eq. (5). They are $C_\varepsilon = C_{\varepsilon\eta} = 0$, $C_\eta = -9/16$, $C_{\varepsilon^2} = 37/16$, $C_{\eta^2} = 271/256$. With this, we have found the generic solution of Eq. (5), up to the second order in both small parameters:

$$
\begin{aligned}
bu = &\cos\varphi+\frac{\varepsilon}{2}\left(3-\cos 2\varphi\right)-\frac{\eta}{16}\left(9\cos\varphi-\cos 3\varphi+12\varphi\sin\varphi\right) \\
&+\frac{\varepsilon^2}{16}\left(37\cos\varphi+3\cos 3\varphi+60\varphi\sin\varphi\right) \\
&+\frac{\eta^2}{256}\big(271\cos\varphi-48\cos 3\varphi+\cos 5\varphi \\
&+384\varphi\sin\varphi-36\varphi\sin 3\varphi-72\varphi^2\cos\varphi\big) \\
&+\frac{\varepsilon\eta}{16}\left(-87+40\cos 2\varphi-\cos 4\varphi+12\varphi\sin 2\varphi\right)\ . \qquad (20)
\end{aligned}
$$

(Note that the coefficients of $\cos\varphi$ in the ε^2 and η^2 terms are corrected with respect to reference [34], where in the solution of Eq. (7) the choice of the constants C_{ε^2} and C_{η^2} was not verified to solve Eq. (5).)

Far away from the lensing object $u = 0$ and $\varphi = \pm\pi/2 \pm \delta\varphi/2$, where the $+\,(-)$ sign is for the light signal in the distant future (past), and $\delta\varphi$ represents the angle with which the light ray is bent by the lensing object with mass m and tidal charge q. In our second-order approach this has the form:

$$
\delta\varphi = \varepsilon\alpha_1+\eta\beta_1+\varepsilon^2\alpha_2+\eta^2\beta_2+\varepsilon\eta\gamma_2+\mathcal{O}\left(\varepsilon^3,\eta^3,\varepsilon\eta^2,\varepsilon^2\eta\right)\ . \qquad (21)
$$

A power series expansion of the solution (20) in the small parameters then gives the coefficients of the above expansion, and the deflection angle becomes:

$$
\delta\varphi = 4\varepsilon-\frac{3\pi}{4}\eta+\frac{15\pi}{4}\varepsilon^2+\frac{105\pi}{64}\eta^2-16\varepsilon\eta\,. \qquad (22)
$$

The first three terms of this expansion were already given in [35] for the Reissner-Nordström black hole. There, however the argument that η is of ϵ^2 order was advanced. In brane-worlds there is no a priori reason for considering only small values of the tidal charge, thus we have computed the deflection angle $\delta\varphi$ containing all possible contributions up to second order in both parameters.

The deflection angle however is given in terms of the Minkowskian impact parameter b. It would be useful to write this in term of the distance of minimal approach $r_{\min}$ as well. The minimal approach is found by inserting the values $u = 1/r_{\min}$ and $\varphi = 0$ in Eq. (20):

$$
r_{\min} = b\left(1-\varepsilon+\frac{1}{2}\eta-\frac{3}{2}\varepsilon^2-\frac{5}{8}\eta^2+2\varepsilon\eta\right)\ . \qquad (23)
$$

Inverting this formula gives to second order accuracy (the small parameters being now $m/r_{\min}$ and $q/r_{\min}^2$):

$$\frac{1}{b}=\frac{1}{r_{\min}}\left(1-\frac{m}{r_{\min}}+\frac{q}{2r_{\min}^2}-\frac{m^2}{2r_{\min}^2}-\frac{q^2}{8r_{\min}^4}+\frac{mq}{2r_{\min}^3}\right) . \tag{24}$$

As the deflection angle consists only of first and second order contributions, the above formula is needed only to first order for expressing $\delta\varphi$ in terms of the minimal approach:

$$\delta\varphi=\frac{4m}{r_{\min}}-\frac{3\pi q}{4r_{\min}^2}+\frac{(15\pi-16)\,m^2}{4r_{\min}^2}+\frac{57\pi q^2}{64r_{\min}^4}+\frac{(3\pi-28)\,mq}{2r_{\min}^3} . \tag{25}$$

The first three terms again agree with the ones given in [35], when $q=Q^2$.

3. Hamiltonian approach

In this section we employ the eikonal method for deriving the light deflection angle. Subsection 3.1 follows the derivation presented in Ref. [26], which in turn is based on Ref. [37]. Instead of the coordinate transformation employed in Ref. [26], in subsection 3.2 we follow a perturbative approach based on a double expansion in both small parameters. This ensures a higher accuracy of the perturbative result, and we recover the deflection angle (25). Some of the technical details are presented in the Appendix.

3.1. Light deflection from the eikonal equation

This derivation starts from the general relativistic eikonal equation (Hamilton-Jacobi equation)

$$g^{ab}\frac{\partial\Psi}{\partial x^a}\frac{\partial\Psi}{\partial x^b}=0 . \tag{26}$$

Here the function Ψ is the rapidly varying real phase of the complex electromagnetic 4-potential A_a =Re$[\mathcal{A}_a\exp(i\Psi)]$ and $k_a=\partial\Psi/\partial x^a$ is the wave vector. The complex amplitude varies only slowly in the geometrical optics (eikonal / high-frequency) approximation: the wavelength is small compared to either of the characteristic curvature radius or the typical length of variation of the optical properties. The normalized version of the complex amplitude is the polarization vector. Light rays are defined as the integral curves of k^a and they are perpendicular to the wave-fronts, the surfaces of constant phase. The eikonal equation then is but the condition that the wave vector is null. The vacuum Maxwell equations further imply that light travels along null geodesics, $k^a\nabla_a k^b=0$ and the polarization vector is perpendicular to the light rays and parallel propagated along them (fore more details see Chapter 1.8. of Ref. [36].

As before, we discuss orbits in the equatorial plane $\theta=\pi/2$. Due to the symmetries of the problem the eikonal function (which is the Hamilton-Jacobi action) can be chosen as

$$\Psi=E\left(-t+b\varphi\right)+\psi_r\left(r\right) \ , \tag{27}$$

where we have employed the relation $L=bE$ derived at the end of subsection 2.1.

The eikonal equation (26) gives for the unknown radial function

$$\psi_r = E\int \mathcal{C}\left(r\right)dr \ , \tag{28}$$

$$\mathcal{C}\left(r\right) = \pm\sqrt{\frac{r^4}{\left(r^2-2b\varepsilon r+b^2\eta\right)^2}-\frac{b^2}{r^2-2b\varepsilon r+b^2\eta}} \ . \tag{29}$$

Here we have employed the definition (8) of the small parameters ε and η and we choose the negative root in front of the square root when r decreases (the photon approaches the lensing object) and the positive root when r increases (the photon has already overpassed the lensing object). This choice assures $d\psi_r\left(r\right)=E\mathcal{C}\left(r\right)dr>0$, regardless whether the photon is approaching or departing from the point of closest approach.

By differentiating Eq. (27) with respect to L we obtain

$$\frac{d\Psi}{dL}=\varphi+\frac{d\psi_r}{dL} \ , \tag{30}$$

however due to Jacobi's Theorem the derivative of the Hamilton-Jacobi action with respect to a canonical constant (in this case L), gives another canonical constant. [Ψ is the generating function of the canonical transformation to pairs of variables trivially obeying the Hamiltonian equations; in the present case L and $d\Psi/dL$ such that $\left(d/d\lambda\right)L=0=\left(d/d\lambda\right)\left(d\Psi/dL\right)$, where λ is a parameter along the trajectory of the photon.]

For later convenience we introduce a new auxiliary variable ϕ by $r=b/\cos\phi$. To zeroth order in the small parameters the relation $\varphi=\phi$ holds, see the remarks at the end of subsection 2.1. As $r>0$ we have $\arccos\left(b/r\right)=\left|\phi\right|$, in other words $\phi=\ \mathrm{sgn}\phi\arccos\left(b/r\right)$. The variable ϕ exists for any $r\geq b$. By evaluating Eq. (30) at two points $r\geq b$ on the trajectory $\varphi\left(\phi\right)$ and forming the difference leads to

$$\varphi\left(\phi_2\right)-\varphi\left(\phi_1\right)=-\left.\frac{d\psi_r}{dL}\right|_{\phi_2}+\left.\frac{d\psi_r}{dL}\right|_{\phi_1} \ . \tag{31}$$

While the photon travels from the infinity to the point of closest approach $r=r_{\min}$ and then back to infinity, the total change in the polar angle φ can be found in a limiting process as

$$\Delta\varphi=-\lim_{\Phi\rightarrow\pi/2}\left(\left.\frac{\partial\psi_r}{\partial L}\right|_{\Phi}-\left.\frac{\partial\psi_r}{\partial L}\right|_{-\Phi}\right) \ . \tag{32}$$

(Here $\Phi\geq0$.)

3.2. Perturbative solution

We expand the radial function ψ_r given by (29) to second order accuracy in both ε and η:

$$\psi_r = L\int \mathcal{C}\left(\phi\right)\frac{\sin\phi}{\cos^2\phi}d\phi \ , \tag{33}$$

$$\begin{aligned}\mathcal{C}\left(\phi\right) &= \sin\phi+\left(\varepsilon-\frac{1}{2}\eta\cos\phi\right)\frac{\left(2-\cos^2\phi\right)\cos\phi}{\sin\phi}\\ &+\frac{1}{2}\left(\varepsilon-\frac{1}{2}\eta\cos\phi\right)^2\frac{\cos^2\phi}{\sin^3\phi}\\ &\times\left[4\left(3-\cos^2\phi\right)\sin^2\phi-\left(2-\cos^2\phi\right)^2\right] \ .\end{aligned} \tag{34}$$

The expression $\mathcal{C}(\phi) \equiv \mathcal{C}(r = b/\cos\phi)$ changes sign with sgnϕ, in accordance with the assumption made in Eq. (29) for the sign in front of the square root.

By the computation presented in the Appendix it is immediate to derive the zeroth, ε, η, ε^2, η^2, $\varepsilon\eta$ order contributions to $\Delta\varphi$ as $(\pi,\, 4,\, -3\pi/4,\, 15\pi/4,\, 105\pi/64,\, -16)$.

To zeroth order we have found that $(\Delta\varphi)_0 = \pi$, thus the path of the photon in the absence of the perturbing object with mass m and tidal charge q is a straight line. Thus the deflection caused by the mass and tidal charge of the lensing object when the photon travels from the infinity to the nearest point $r = r_{\min}$ and then back to the infinity is given by $\delta\varphi = \Delta\varphi - \pi$.

In consequence the term-by-term computation by the limiting process (32) reproduces exactly the deflection angle (22), derived earlier in a Lagrangian approach.

4. Solar system constraints

The most important difference in comparing Eq. (22) to Eq. (27) of Ref. [26] is the presence of the η-term in our result, which turns out to be the dominant contribution to the deflection angle caused by the tidal charge. This is ε^{-1} times larger than the $\varepsilon\eta$ mixed term.

This implies that the constraints on the tidal charge derived in Ref. [26], as imposed by the measurements of the deflection of light by the Sun should be re-evaluated. For this we follow the logic of Ref [26], but apply the observational constraint to the η term (rather than $\varepsilon\eta$). Long baseline radio interferometry measurements [38]-[39] give $\delta\varphi = \delta_\varepsilon\varphi\,(1+\xi)$, with $\xi < \xi_{\max} = \pm 0.0017$. By assuming that the dominant deviation from the (first order) Schwarzschild value is due to the tidal charge, we obtain: $\delta_\varepsilon\varphi\xi_{\max} = (\delta_\eta\varphi)_{\max}$, thus $16\varepsilon\xi_{\max} = 3\pi\,(-\eta)_{\max}$ or $16mb\xi_{\max} = 3\pi\,(-q)_{\max}$. With the mass of the Sun $m = M_\odot = 1476.685$m and the smallest possible impact parameter (equal to the closest possible approach $r_{\min} = R_\odot = 695990$ km), in the first order approximation employed here:

$$|q|_{\max} = \frac{16\,|\xi_{\max}|}{3\pi} M_\odot R_\odot = 2966\mathrm{km}^2 \ . \tag{35}$$

The junction conditions of the tidal charged brane black hole with a star of uniform density ρ (which dominates over the tidal contribution to the energy density), applied for the Sun give a negative tidal charge [25]:

$$q_\odot = -\frac{3M_\odot R_\odot \rho_\odot}{\lambda} \ . \tag{36}$$

From $-q_\odot \leq |q|_{\max}$ we find

$$\lambda \geq \frac{3M_\odot R_\odot \rho_\odot}{|q|_{\max}} = \frac{9\pi\rho_\odot}{16\,|\xi_{\max}|} = 1464.066\frac{\mathrm{g}}{\mathrm{cm}^3} = 6.310\cdot 10^{-3}\mathrm{MeV}^4 \ . \tag{37}$$

The constraint on the brane tension from Solar System measurements is therefore 5 orders of magnitude stronger than derived in Ref. [26], however still far weaker than all other constraints ($\lambda \geq 138.59$ TeV^4 from table-top experiments [40], [41], $\lambda \geq 1$MeV^4 from nucleosynthesis [42] and $\lambda \geq 5\times 10^8$ MeV^4 from neutron stars [25]).

5. Concluding remarks

In this paper we have computed the light deflection angle due to a tidal charged brane black hole / naked singularity / star (depending on q), up to second order in the two

small parameters ε and η, related to the mass and tidal charge of the lensing object. We have carried on this task by two distinct methods and obtained identical results.

The first method relies on a Lagrangian, while the second on a Hamilton-Jacobi approach. The latter was first applied in Ref. [26], however that calculation focused only on the first order correction in q of the Schwarzschild deflection angle (thus an $\varepsilon\eta$-contribution in our terminology), given in their Eq. (27).

In comparison, besides the Schwarzschild contribution ε, our result (22) for the light deflection angle contains the second order Schwarzschild correction ε^2, the first and second order tidal contributions η and η^2, finally the mixed contribution $\varepsilon\eta$. The latter turns out to be twice the value given in Ref. [26], where the expansion was not performed everywhere to this order (for example in the Jacobian of the transformation involved there). As a consequence of these improvements we have strengthened the limit imposed on the brane tension by Solar System measurements by 5 orders of magnitude.

As already remarked in [43], the electric charge of the Reissner-Nordström black hole decreases the deflection angle, as compared to the Schwarzschild case. The same holds true for a *positive* tidal charge. If the condition $16mr_{\min} = 3\pi q$ is obeyed, the first order contributions to the deflection angle cancel (there is no deflection to first order) and the three second order terms of $\delta\varphi$ presented in this paper give the leading effect to weak lensing.

Furthermore, $16mr_{\min} < 3\pi q$ could be obeyed, leading to a *negative* deflection angle (to first order). That would mean that rather than magnifying distant light sources, such a lensing object will demagnify them.

By contrast, a negative tidal charge can considerably increase the lensing effect. Such negative tidal charged brane black holes arise naturally as exteriors of static brane stars composed of ideal fluid with constant density [25]. Therefore a negative tidal charge could be responsible at least for part of the lensing effects attributed at present to dark matter. However in line with the Solar System constraint (35) imposed in this paper on the tidal charge of the Sun, we do not expect these contributions to be very large.

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Appendix A. The change in the polar angle

The radial contribution (34) to the eikonal function can be written as the series

$$\psi_r(\phi) = L\left(I_0 + \varepsilon I_\varepsilon + \eta I_\eta + \varepsilon^2 I_{\varepsilon^2} + \eta^2 I_{\eta^2} + \varepsilon\eta I_{\varepsilon\eta}\right) \ , \tag{A.1}$$

with

$$\begin{aligned} I_0(\phi) &= \tan\phi - \phi \ , \\ I_\varepsilon(\phi) &= \ln\left(\frac{1+\sin\phi}{1-\sin\phi}\right) - \sin\phi \end{aligned}$$

$$
\begin{aligned}
&= 2\mathrm{sgn}\phi \,\operatorname{arccosh}\left(\frac{1}{\cos\phi}\right) - \sin\phi \ , \\
I_{\eta}(\phi) &= \frac{1}{4}\sin\phi\cos\phi - \frac{3\phi}{4} \ , \\
I_{\varepsilon^2}(\phi) &= \frac{15}{4}\phi + \left(3\cos^2\phi - 1\right)\frac{\cot\phi}{4} \ , \\
I_{\eta^2}(\phi) &= \frac{35}{64}\phi + \left(6\cos^4\phi - 33\cos^2\phi + 35\right)\frac{\cot\phi}{64} \ , \\
I_{\varepsilon\eta}(\phi) &= \frac{8\cos^2\phi - \cos^4\phi - 8}{2\sin\phi} \ .
\end{aligned}
\tag{A.2}
$$

As all of these terms are antisymmetric, the radial contribution to the eikonal function is also antisymmetric, $\psi_r(-r) = -\psi_r(r)$.

Besides the explicit global factor L, the expression $\psi_r(\phi)$ also depends on L through $\varepsilon = mb^{-1}$, $\eta = qb^{-2}$ and $\phi = \mathrm{sgn}\phi \arccos(b/r)$, as $b = L/E$. However the products $L\varepsilon$ and $L^2\eta$ are independent of L. In consequence the derivative $d\psi_r(\phi)/dL$ can be calculated as

$$
\begin{aligned}
\frac{d}{dL}\psi_r(\phi) &= \frac{d}{dL}(LI_0) + (L\varepsilon)\frac{d}{dL}I_{\varepsilon} + \left(L^2\eta\right)\frac{d}{dL}\left(L^{-1}I_{\eta}\right) \\
&+ (L\varepsilon)^2\frac{d}{dL}\left(L^{-1}I_{\varepsilon^2}\right) + \left(L^2\eta\right)^2\frac{d}{dL}\left(L^{-3}I_{\eta^2}\right) \\
&+ (L\varepsilon)\left(L^2\eta\right)\frac{d}{dL}\left(L^{-2}I_{\varepsilon\eta}\right) \ .
\end{aligned}
\tag{A.3}
$$

The term-by-term computation, by employing the identity

$$
\frac{d\phi}{dL} = -L^{-1}\cot\phi \ , \tag{A.4}
$$

gives

$$
\begin{aligned}
\frac{d}{dL}(LI_0) &= -\phi \ , \\
L\frac{d}{dL}I_{\varepsilon} &= \frac{-2}{\sin\phi} + \frac{\cos^2\phi}{\sin\phi} \ , \\
L^2\frac{d}{dL}\left(L^{-1}I_{\eta}\right) &= \frac{3\phi}{4} - \frac{\cos^2\phi - 3}{4\sin\phi}\cos\phi \ , \\
L^2\frac{d}{dL}\left(L^{-1}I_{\varepsilon^2}\right) &= -\frac{15\phi}{4} - \frac{9\cos^4\phi - 26\cos^2\phi + 15}{4\sin^3\phi}\cos\phi \ , \\
L^4\frac{d}{dL}\left(L^{-3}I_{\eta^2}\right) &= -\frac{105\phi}{64} \\
&\quad - \frac{6\cos^6\phi + 21\cos^4\phi - 105\cos^2\phi + 70}{64\sin^3\phi}\cos\phi \ , \\
L^3\frac{d}{dL}\left(L^{-2}I_{\varepsilon\eta}\right) &= \frac{8}{\sin\phi} + \frac{\cos^4\phi + 6\cos^2\phi - 24}{2\sin\phi}\cos^2\phi \ .
\end{aligned}
\tag{A.5}
$$

As the last term of each expression contains the factor $\cos\phi$, only the first terms will survive the limiting process described in subsection 3.1. The respective limits are given in subsection 3.2.

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